SPRING 2025 MATH 590: QUIZ 8 SOLUTIONS

Name:

1. State the spectral theorem for real symmetric matrices. Be sure to define all terms used in the statement. (3 points)

Solution. The spectral theorem for real symmetric matrices states: Given a symmetric matrix A over the real numbers, there exists an orthogonal matrix Q such that $Q^{-1}AQ$ is a diagonal matrix. Alternatively: Every real symmetric matrix is orthogonally diagonalizable.

Definitions: An $n \times n$ matrix A is symmetric, if $A^t = A$. An $n \times n$ matrix Q is orthogonal if $Q^{-1} = Q^t$, equivalently, the columns of A form an orthonormal basis for $\mathbb{R}^n$. For the alternative version: A is orthogonally diagonalizable, if there is an orthogonal matrix Q such that $Q^{-1}AQ$ is diagonal.

2. State one of the key facts about symmetric matrices that enable one to prove the spectral theorem for symmetric matrices. (2 points)

Solution. The two key facts are: (i) Any real symmetric matrix has its eigenvalues in $\mathbb{R}$ and (ii) If $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$ with $\lambda_1 \neq \lambda_2$, then $\langle v_1, v_2 \rangle = 0$, i.e., eigenvectors corresponding to distinct eigenvalues are orthogonal.

3. Find an orthogonal 3×3 matrix that diagonalizes $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Then check your answer. (5 points)

Solution. $p_A(x) = \begin{vmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{vmatrix} = x^3 - 3x - 2 = (x+1)^2(x-2)$. Thus -1 (with algebraic multiplicity two) and 2 are the eigenvalues of A.

E_{-1} is the null space of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which has basis $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

E_2 is the nullspace of $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, which has basis $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Note that v_1, v_3 and v_2, v_3 are orthogonal, but v_1, v_2 are not orthogonal.

Using Gram-Schmidt, we orthogonalize v_1, v_2: We take $w_1 = v_1$ and

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix}.$$

We can get rid of the denominators, by taking $w_3' = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$. Normalizing the vectors w_1, w_2, w_3', we get

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, u_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, u_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We now take $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$, which is an orthogonal matrix diagonalizing A. To verify this, we check $AQ = QD$, where $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

$$AQ = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{3}} \end{pmatrix}.$$

$$QD = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{3}} \end{pmatrix}.$$